LESSON

18 Antidifferentiation

The Fundamental Theorem of Calculus shows that the area under the curve, $\int_a^b f(x)\,dx$, can be calculated with a function $g(x)$ whose derivative is $g'(x) = f(x)$:

$$\int_a^b f(x)\,dx = [g(x)]_a^b = g(b) - g(a)$$

Because of this, the symbol $\int$, without the limits of integration, is used to represent the opposite of taking the derivative. An integral like $\int_a^b f(x)\,dx$ is called a *definite integral* because it represents a definite area. An integral like $\int f(x)\,dx$ is called an *indefinite integral* because it represents another function.

Thus, $\int f(x)\,dx$ means the *antiderivative* of $f(x)$ or "the function whose derivative is $f(x)$." For example, $\int 2x\,dx$ asks "whose derivative is $2x$?" This could be x^2 because $\frac{d}{dx}(x^2) = 2x$. However, it could also be $x^2 + 5$ because $\frac{d}{dx}(x^2 + 5) = 2x$ as well. In fact, because the derivative of a constant is zero, $\int 2x\,dx$ could be x^2 plus any constant. Therefore, we write $\int 2x\,dx = x^2 + c$ where c is any constant.

Bracket Note

The brackets $[\ldots]_a^b$ are just a way of keeping track of the *limits of integration* a and b before they are plugged into $g(x)$ and subtracted.

Again, because mathematicians are lazy, we usually simply write $\int 2x\,dx = x^2 + c$ and assume that everyone knows that the c stands for "some constant." In many ways, the "plus c" is the trademark of the indefinite integral because every problem that begins with $\int (\ldots.)\,dx$ ends with $+\,c$.

If we are dealing with a definite integral like $\int_3^5 2x\,dx$, then it does not matter what constant we use. For example:

$$\int_3^5 2x\,dx = [x^2 + c]_3^5$$

$$= (5^2 + c) - (3^2 + c)$$

$$= 25 + c - 9 - c = 16$$

The "plus c" will always cancel out in the subtraction, so we may as well simply use $c = 0$ and write:

$$\int_3^5 2x\,dx = [x^2]_3^5 = 5^2 - 3^2 = 25 - 9 = 16$$

Example

Use $\frac{d}{dx}(x^3 + 10x^2 + 3x) = 3x^2 + 20x + 3$ to evaluate $\int (3x^2 + 20x + 3)\,dx$ and $\int_1^2 (3x^2 + 20x + 3)\,dx$.

Solution

Because $\frac{d}{dx}(x^3 + 10x^2 + 3x) = 3x^2 + 20x + 3$, we know that:

$$\int (3x^2 + 20x + 3)\,dx = x^3 + 10x^2 + 3x + c$$

Similarly,

$$\int_1^2 (3x^2 + 20x + 3)\,dx = [x^3 + 10x^2 + 3x]_1^2$$

$$\int_1^2 (3x^2 + 20x + 3)\,dx =$$

$$((2)^3 + 10\cdot(2)^2 + 3\cdot(2)) -$$
$$((1)^3 + 10\cdot(1) + 3\cdot(1))$$

$$\int_1^2 (3x^2 + 20x + 3)\,dx =$$

$$8 + 40 + 6 - (1 + 10 + 3) = 54 - 14 = 40$$

The general rules for antiderivatives are fairly simple. To take the derivative of $f(x) = x^5$, we first multiply by the exponent 5, and then we subtract one from the exponent. Thus, $f'(x) = 5x^4$.

To antidifferentiate $\int 5x^4 dx$, we must do the exact opposite of this process. First, we add one to the exponent, and then we divide the result by the new

Verification Hint

You can verify your answer by taking its derivative. If the derivative of your answer is what you were trying to integrate, then you are correct.

The derivative of $\frac{2}{3}x^{\frac{3}{2}} + c$ is $\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}} + c\right) = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{1}{2}} + 0 = x^{\frac{1}{2}} = \sqrt{x}$. This verifies that $\int \sqrt{x}dx = \frac{2}{3}x^{\frac{3}{2}} + c$.

exponent. Thus, $\int 5x^4dx = \frac{5x^{4+1}}{4+1} + c = x^5 + c$. In general, we write:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1$$

Example

Evaluate $\int x^7 dx$.

Solution

$$\int x^7 dx = \frac{x^{7+1}}{7+1} + c = \frac{1}{8}x^8 + c$$

Example

Evaluate $\int \sqrt{x}dx$.

Solution

$$\int \sqrt{x}dx = \int x^{\frac{1}{2}}dx$$

$$= \frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}x^{\frac{3}{2}} + c$$

Example

Evaluate $\int_0^2 x^3 dx$.

Solution

$$\int_0^2 x^3 dx = \left[\frac{1}{4}x^4\right]_0^2$$

$$\int_0^2 x^3 dx = \frac{1}{4}\cdot 2^4 - \frac{1}{4}\cdot 0^4$$

$$= \frac{1}{4}\cdot 16 - \frac{1}{4}\cdot 0 = 4 - 0 = 4$$

▶ Practice

Evaluate the following integrals.

1. $\int x^5 dx$

2. $\int x^{12} dx$

3. $\int u^6 du$

4. $\int_0^6 x^2\, dx$

5. $\int_1^9 x\,dx$

6. $\int t^{-3} dt$

7. $\int_{-1}^{2} t^5\,dt$

8. $\int x^{\frac{5}{3}}dx$

9. $\int \sqrt[3]{x}dx$

10. $\int \sqrt[3]{u}du$

11. $\int 5dx$

12. $\int 5dt$

13. $\int_{-1}^{4} 8\,dx$

14. $\int_{2}^{4} \frac{1}{x^2}dx$

Just as with derivatives, constants can stand aside, and the terms of sums can be dealt with separately.

Example

Evaluate $\int 5x^2\,dx$.

Solution

$$\int 5x^2\,dx = 5\left(\frac{1}{3}x^3\right) + c$$

$$\int 5x^2\,dx = \frac{5}{3}x^3 + c$$

Example

Evaluate $\int (2t^3 - 8t + 7)\,dt$.

Solution

$$\int (2t^3 - 8t + 7)\,dt = 2\cdot\frac{1}{4}t^4 - 8\cdot\frac{1}{2}t^2 + 7t + c$$

$$\int (2t^3 - 8t + 7)\,dt = \frac{1}{2}t^4 - 4t^2 + 7t + c$$

Example

Evaluate $\int_1^4 \left(6\sqrt{x} - \frac{8}{x^5}\right)dx$.

Solution

It always helps to write everything in exponential form.

$$\int_1^4 \left(6\sqrt{x} - \frac{8}{x^5}\right)dx = \int_1^4 (6x^{\frac{1}{2}} - 8x^{-5})\,dx$$

$$\int_1^4 \left(6\sqrt{x} - \frac{8}{x^5}\right)dx = \left[6\cdot\left(\frac{2}{3}x^{\frac{3}{2}}\right) - 8\left(\frac{x^{-4}}{-4}\right)\right]_1^4$$

$$\int_1^4 \left(6\sqrt{x} - \frac{8}{x^5}\right)dx = \left[4x^{\frac{3}{2}} + \frac{2}{x^4}\right]_1^4$$

$$\int_1^4 \left(6\sqrt{x} - \frac{8}{x^5}\right)dx =$$

$$\left(4(8) + \frac{2}{256}\right) - \left(4 + \frac{2}{1}\right) = 26 + \frac{1}{128}$$

▶ Practice

Evaluate the following integrals.

15. $\int 9x^4\,dx$

16. $\int 8u^2\,du$

17. $\int (x - \sqrt{x})\,dx$

18. $\int (6x^2 - 10x + 5)\,dx$

19. $\int_0^2 12x^3\,dx$

20. $\int_1^2 (3x^2 + 4)\,dx$

21. $\int_2^7 (6x - 4)\,dx$

22. $\int (3t^{11} + 9t^2 + t)\,dt$

23. $\int_0^3 (1 - t^2)\,dt$

24. $\int (8x^3 + 10x^2 - 4x + 2)\,dx$

25. $\int_0^2 (10u^4 - 4u + 1)\,du$

26. $\int_1^9 12\sqrt{x}\,dx$

27. $\int \frac{4}{x^3}\,dx$

28. $\int (3x^{\frac{10}{3}} - 8x^{\frac{4}{3}})\,dx$

The integrals of e^x, $\sin(x)$, and $\cos(x)$ follow directly from their derivatives:

$$\int e^x\,dx = e^x + c \text{ because } \frac{d}{dx}(e^x) = e^x$$

$$\int \cos(x)\,dx = \sin(x) + c \text{ because}$$
$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\int \sin(x)\,dx = -\cos(x) + c \text{ because}$$
$$\frac{d}{dx}(-\cos(x)) = \sin(x)$$

The integral of $\ln(x)$ will have to wait until Lesson 20, though we can use the fact that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ right now. We are inclined to say that $\int \frac{1}{x}\,dx = \ln(x) + c$, but this is not entirely correct. The derivative of $\ln(-x)$ is $\frac{d}{dx}(\ln(-x)) = \frac{1}{-x}\cdot\frac{d}{dx}(-x) = \frac{1}{-x}\cdot(-1) = \frac{1}{x}$ as well. It does not matter if the x inside the natural logarithm is positive or negative, so we can generalize with the absolute value $|x|$.

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

Incidentally, this nicely fills a hole in an earlier formula:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ if } n \neq -1$$

and if $n = -1$ then

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + c$$

Example

Evaluate $\int (3\sin(x) + 5\cos(x))\,dx$.

Solution

$$\int (3\sin(x) + 5\cos(x))\,dx =$$

$$-3\cos(x) + 5\sin(x) + c$$

Example

Evaluate $\int_0^1 (3t^2 - 5e^t)\,dt$.

Solution

$$\int_0^1 (3t^2 - 5e^t)\,dt = [t^3 - 5e^t]_0^1$$

$$\int_0^1 (3t^2 - 5e^t)\,dt = (1^3 - 5e^1) - (0^3 - 5e^0)$$

$$\int_0^1 (3t^2 - 5e^t)\,dt = 1 - 5e + 5 = 6 - 5e$$

Example

Evaluate $\int \left(x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\right)dx$.

Solution

$$\int \left(x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\right)dx =$$

$$\int (x^2 + x^1 + x^0 + x^{-1} + x^{-2})\,dx$$

$$\int \left(x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\right)dx =$$

$$\frac{1}{3}x^3 + \frac{1}{2}x^2 + x^1 + \ln|x| - x^{-1} + c$$

$$\int \left(x^2 + x + 1 + \frac{1}{x} + \frac{1}{x^2}\right)dx =$$

$$\frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \ln|x| - \frac{1}{x} + c$$

▶ Practice

Evaluate the following integrals.

29. $\int (x^2 - 5\cos(x))\,dx$

30. $\int (3e^x + 2x^3)\,dx$

31. $\int \frac{1}{u}\,du$

32. $\int (\theta + 2\sin(\theta))\,d\theta$

33. $\int (\sin(x) + 2e^x)\,dx$

34. $\int_0^1 (x + e^x)\,dx$

35. $\int_1^e \frac{4}{x}\,dx$

36. $\int_0^{\frac{\pi}{4}} (8\cos(x))\,dx$